

Some qualitative problems in network dynamics*

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Abstract

This paper addresses analytical aspects of deterministic, continuous-time dynamical systems defined on networks. The goal is to model and analyze certain phenomena which must be framed beyond the context of networked dynamical systems, understood as a set of interdependent dynamical systems defined on the nodes of a (possibly evolving) graph. In order to advocate for a more flexible approach to the study of network dynamics, we tackle some qualitative problems which do not fall in this working scenario. First, we address a stability problem on a network of heterogeneous agents, some of which are of dynamic nature while others just impose restrictions on the system behavior. Our second context assumes that the network is clustered, and we address a two-level stability problem involving the dynamics of both individual agents and groups. The aforementioned problems exhibit lines of non-isolated equilibria, and the analysis implicitly assumes a positiveness condition on the edge weights; the removal of this restriction complicates matters, and our third problem concerns a graph-theoretic characterization of the equilibrium set in the dynamics of certain networks with positive and negative weights, the results applying in particular to signed graphs. Our approach combines graph theory and dynamical systems theory, but also uses specific tools coming from linear algebra, including e.g. Geršgorin discs or Maxwell-type determinantal expansions. The results are of application in social, economic, and flow networks, among others, and are also aimed at motivating further research.

Keywords: graph; digraph; signed graph; hypergraph; network; dynamical system; stability; equilibrium; differential-algebraic equation; flow; social network; economic network.

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1 Introduction

The interactions between dynamical systems and networks have attracted the interest of many researches in the last decade. Needless to say, the study of dynamics taking place

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on graph-theoretic structures has a long tradition in different application fields, including nonlinear circuits, water and gas networks, neural nets, flows, coupled dynamical systems and, in particular, coupled oscillators and synchronization, etc., to name but a few. In the framework of what is now known as *network science*, dynamics is one of the key research directions. A great deal of research within this interdisciplinary field is directed to the dynamics of state variables defined on the nodes of a network; the topics involved include networks of coupled dynamical systems [28, 39, 49, 52], consensus and agreement protocols [40, 43, 50, 51], controllability and related aspects [34, 35, 43], epidemics [16, 39], diffusion of information, innovations, etc. [7, 13], dynamics of social networks [13, 25, 36] or evolutionary dynamics [2, 32], among others.

Much attention has also been paid to the study of network growth [2, 6, 41, 42], percolation [38] and, more generally, to the analysis of the dynamic changes of the network itself (cf. [2, 39, 47]). These dynamic changes of the network structure have been studied in the last decades both in a deterministic and in a probabilistic context. The original ideas regarding these *evolving networks* [2] can be traced back to the seminal work of Erdős and Rényi on the evolution of random graphs [17, 18]. Find more references in [2, 38, 39]. In a deterministic, continuous-time context, the changes in the structure of a network modelled as a weighted simple graph can be naturally described as a dynamical system in which the state variables are the graph weights or, equivalently, the adjacency matrix, in the understanding that a vanishing weight w_{ij} models the absence of an edge between nodes i and j . This dynamical system can be defined as a one-parameter group of transformations of a given state space, or as a system of differential equations. Find details in [47].

So-called *coevolutionary* processes involve simultaneously dynamics taking place on a network and changes in the network structure; cf. [19, 20, 25, 29, 36] and references therein. Indeed, it is natural to consider networks in which dynamical changes in the nodes coexist with the evolution of the network itself. With different names, these coevolutionary processes have been addressed in several application fields, including neural network training [5, 24, 46], social network analysis [25, 36], economic models [29], etc. See also [19, 20] and the bibliography therein.

In spite of the large amount of literature addressing dynamical processes on networks, it seems that in a continuous-time, deterministic context, general approaches to the formulation and analysis of network dynamics are somehow too restrictive, being focused on one of just three paradigms: a) systems in which the dynamics takes place only on the network nodes; b) dynamics of the network itself; and c) coevolution processes combining both. The references cited above support this point of view. Certainly, in specific fields, dynamical systems on networks arise beyond these categories; flow networks or nonlinear electrical circuits, which systematically exhibit state variables not supported on the network nodes, are examples of this. But a general approach, studying abstract dynamics on networks beyond the three settings indicated above, seems to be lacking in the literature.

This paper is intended to be a step in this direction, addressing certain dynamical problems which cannot be framed in the working scenarios mentioned above. First, models for

network dynamics formulated in terms of a set of dynamical systems defined on the nodes of a graph somehow assume a homogeneous nature on all agents. By contrast, we address in Section 2 a stability problem involving a potential-driven flow on a network of heterogeneous agents, some of which are of dynamic nature while others just impose restrictions on the system behavior; this naturally leads to a differential-algebraic formalism. In Section 3 we will examine clustered networks, addressing a two-level stability problem which involves the dynamics of both individual agents and groups. Lines of non-isolated equilibria will naturally arise in the two problems just mentioned; the one-dimensional nature of the equilibrium set is implicitly related to a positiveness condition on the edge weights. In (possibly nonlinear) flow problems on networks which may include negative weights, the structure of the equilibrium set becomes more involved: in Section 4 we will examine such structure without the aforementioned positiveness assumption, applying in particular the results to flow networks on signed graphs. Finally, Section 5 compiles some concluding remarks.

2 Heterogeneous agents and constrained dynamics

Along the terms presented in the Introduction, the “dynamics on nodes” scenario for modelling continuous-time, deterministic network dynamics assumes that the dynamical process can be described by an explicit ODE such as

$$x' = f(x), \tag{1}$$

where x is the state vector, which is defined by a set of variables supported on the network nodes. The topology of the network is assumed to be comprised on the vector field f and, for the sake of simplicity, we are ignoring control variables, explicit time dependences associated with system inputs, etc. A very common perspective assumes that the network edges model a coupling between a set of (otherwise independent) dynamical systems defined on the network nodes (see e.g. [28, 39, 49, 52]), in a way such that only the states of adjacent systems are visible by each node. This means that the differential equations above read as

$$x'_i = f_i(x_i, x_{j_1}, \dots, x_{j_{d(i)}}), \tag{2}$$

for the i -th node ($i = 1, \dots, n$); here x_i is the set of state variables defined on node i , and $j_1, \dots, j_{d(i)}$ are the nodes adjacent to i . Note that for a fully connected (complete) network, (2) makes no difference to (1) (this is the case for instance in the Kuramoto model; see e.g. [49]).

Needless to say, the setting described above has a very broad application scope, as the literature discussed in Section 1 shows. However, in other cases this may be too restrictive a framework to model certain dynamical processes on networks. In particular, systems (1) and (2) do not capture the eventual existence of constraints restricting the whole dynamics of the network. These constrained dynamical systems are better framed in the theory of differential-algebraic equations (DAEs; cf. [12, 30, 31, 45]).

In this context, we focus in this section on networks in which two different sets of nodes (agents) coexist; the behavior of some of them are defined by explicit dynamical systems, whereas the others do not involve dynamics explicitly but are governed, instead, by algebraic (non-differential) restrictions. Letting y stand for the dynamic variables, which model the states of agents with explicit dynamics, and z for the algebraic variables, which correspond to agents without explicit dynamics, these systems can be written as a so-called semiexplicit DAE

$$y' = f(y, z) \tag{3a}$$

$$0 = g(y, z), \tag{3b}$$

where f and g capture the topology of the network; an example can be found in (11) below.

2.1 Potential-driven flows and the graph Laplacian

To fix ideas, consider a simple connected network in which a given resource or commodity flows among a set of n agents located at the network nodes; here “simple” means that the network has neither parallel edges nor self-loops. Let us first assume that all agents collect this commodity, and denote the amount of the collected resource at node i by x_i . The flowrate (or flow) of this resource at a given edge j , connecting nodes i and k , will be denoted by u_j . Every edge is endowed with a reference direction; if the j -th edge is directed say from i to k , $u_j > 0$ (resp. $u_j < 0$) means that the resource flows from i to k (resp. from k to i). This drives the problem to the context of directed graphs, although it is worth emphasizing that all the results are independent of the choice of directions. Denote $x = (x_1, \dots, x_n)$, $u = (u_1, \dots, u_m)$, where m is the number of edges.

With these reference directions, the entries of the incidence matrix $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ of the resulting digraph read as

$$a_{ij} = \begin{cases} 1 & \text{if edge } j \text{ leaves node } i \\ -1 & \text{if edge } j \text{ enters node } i \\ 0 & \text{if edge } j \text{ is not incident with node } i. \end{cases} \tag{4}$$

The continuity equations at the nodes can be then written as

$$x' = -Au, \tag{5}$$

since the increase x'_i of the stored resource at node i equals the net flow entering the node, that is, $-A_i u$ with the sign convention above and A_i standing for the i -th row of A .

System (5) is underdetermined. The way in which the flowrates u interact with the node state variables x may be defined from different perspectives; for instance, in a game-theoretic setting the agents’ strategies would define the flows, or in the framework of control theory u might be designed as to achieve a given goal. Here we will simply assume that u is designed

in order to get a fair (equal) distribution of the commodity among all agents. This can be achieved by setting the following potential-driven definition of the flowrates:

$$u = A^\top x, \quad (6)$$

which is just a redistribution law in which the flow from i to k equals $x_i - x_k$. It is a trivial matter to recast (5)-(6) as

$$x' = -\mathbb{L}x \quad (7)$$

where $\mathbb{L} = AA^\top$ is the graph Laplacian matrix. Note that (7) has the form depicted in (2), as a consequence of the identity $\mathbb{L} = -C + D$, where C is the adjacency matrix of the graph (without directions) and D is a diagonal matrix whose i -th entry is the degree of node i . The dynamics of (7) has been analyzed in the context of so-called *consensus* protocols [40, 43, 50, 51], and the state variables x may be checked to converge to a common value which is the arithmetic mean of the initial values $x_1(0), \dots, x_n(0)$. This means that the resource is indeed redistributed among all the agents in a way such that all of them asymptotically store the same quantity.

2.2 Heterogeneous agents

We now drive our attention to a different problem; given again a simple connected network, assume that only some agents, say type-1 agents (which w.l.o.g. are assumed to be those from 1 to r , with $1 \leq r < n$) accumulate the aforementioned resource, again in an amount x_i for the i -th agent. The continuity equations for type-1 agents read as above,

$$x'_i = -A_i u, \quad i = 1, \dots, r. \quad (8)$$

The other agents, say type-2 ones (those numbered from $r + 1$ to n) are different and do not collect the commodity; instead, they are however intermediaries among type-1 agents. Therefore, the continuity equations at type-2 agents are

$$0 = A_i u, \quad i = r + 1, \dots, n. \quad (9)$$

Finally, we assume that the flowrates are still given by (6); note that by means of the variables x_i type-2 agents now simply set up a reference value for the flowrates u . Denote $y = (x_1, \dots, x_r)$, $z = (x_{r+1}, \dots, x_n)$ (and, for later use, $y_i = x_i$, $z_i = x_{i+r}$) and split A as

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},$$

where A_1 (resp. A_2) is the submatrix of A defined by the first r rows (resp. last $n - r$ rows). The boldface subscripts in A_1 and A_2 are meant to distinguish these submatrices from the first and second row of A , to be written as A_1 , A_2 . With this notation, the network dynamics is modelled by the DAE

$$y' = -A_1 A^\top x \quad (10a)$$

$$0 = A_2 A^\top x. \quad (10b)$$

Splitting $x = (y, z)$ in the right-hand side, and denoting $\mathbb{L}_{ij} = A_i A_j^\top$ for ease of notation, this DAE can be rewritten as

$$y' = -\mathbb{L}_{11}y - \mathbb{L}_{12}z \quad (11a)$$

$$0 = \mathbb{L}_{21}y + \mathbb{L}_{22}z. \quad (11b)$$

Note that this constrained system has the differential-algebraic form displayed in (3).

Theorem 1 describes the dynamics of the DAE (11); in particular, it shows that the scheme defined above redistributes the commodity in a way such that all type-1 agents asymptotically accumulate the same amount of the resource. Note also that there is a somewhat unusual phenomenon, namely the existence of a line of non-isolated equilibria. We use from DAE theory the notion of a *consistent initial value*, which is a value of $(y(0), z(0))$ which satisfies the constraints (11b) (cf. [12, 30, 31, 45]).

Theorem 1. *In the setting described above, the following assertions hold for the DAE (11).*

- (a) *The constraint (11b) specifies an r -dimensional solution space. Its (transversal) intersection with the hyperplanes $\sum_{i=1}^r y_i = k \in \mathbb{R}$ defines a foliation of the dynamics by a family of $(r - 1)$ -dimensional invariant spaces.*
- (b) *All equilibrium points are located in the line $y_1 = y_2 = \dots = y_r = z_1 = \dots = z_{n-r}$, which is transversal to the aforementioned invariant spaces if $r \geq 2$.*
- (c) *The trajectory $(y(t), z(t))$ emanating from a given consistent initial value $(y(0), z(0))$ converges exponentially to the equilibrium*

$$y_i = z_j = \frac{y_1(0) + \dots + y_r(0)}{r}, \quad i = 1, \dots, r, \quad j = 1, \dots, n - r.$$

Proof.

(a) A well-known property in digraph theory states that any proper subset of the rows of the incidence matrix A of a connected digraph has maximal rank. This implies that the matrix A_2 , defined by the last $n - r$ rows of A , has maximal row rank (note that $r \geq 1$). In turn, this means that $\mathbb{L}_{22} = A_2 A_2^\top$ is a non-singular matrix; indeed, provided that $A_2 A_2^\top v = 0$ we get $v^\top A_2 A_2^\top v = 0$, and then $A_2^\top v = 0$ yields $v = 0$. Since $A_2 A_2^\top$ has order $n - r$, it follows that (11b) specifies an r -dimensional linear space.

This space is filled by solutions of the DAE, which are defined from those of the explicit ODE

$$y' = -(\mathbb{L}_{11} - \mathbb{L}_{12} \mathbb{L}_{22}^{-1} \mathbb{L}_{21})y \quad (12)$$

by means of the additional relation

$$z = -\mathbb{L}_{22}^{-1} \mathbb{L}_{21}y. \quad (13)$$

That (11b) intersects transversally the hyperplanes $\sum_1^r y_i = k \in \mathbb{R}$ can be seen again as a consequence of the fact that $\mathbb{L}_{22} = A_2 A_2^\top$ is non-singular, since the coefficient matrix of the set of linear equations defining the intersection reads as

$$\begin{pmatrix} \mathbb{1}_r^\top & 0 \\ \mathbb{L}_{21} & \mathbb{L}_{22} \end{pmatrix},$$

where $\mathbb{1}_r^\top \in \mathbb{R}^{1 \times r}$ has all entries equal to one. This matrix is easily seen to have maximal row rank, meaning that the intersection is indeed transversal and hence defines an $(r-1)$ -dimensional space, for any fixed k .

Denoting by $\mathbb{1}$ the vector of 1's in $\mathbb{R}^{n \times 1}$, the invariance of these spaces follows from the identity $\mathbb{1}^\top A = 0$ (expressing the fact that the sum of all rows of the incidence matrix A vanishes), which readily yields $\mathbb{1}^\top \mathbb{L} = 0$ since $\mathbb{L} = A A^\top$. Recasting (11b) as $-\mathbb{L}x = 0$, from (11) one gets $\mathbb{1}_r^\top y' = 0$, meaning that $y_1 + \dots + y_r$ indeed remains constant along trajectories.

(b) The equations $y_1 = y_2 = \dots = y_r = z_1 = \dots = z_{n-r}$ define a line of equilibria because of the identity $A^\top \mathbb{1} = 0$, which means that $\mathbb{1}$ spans $\ker \mathbb{L} = \ker A A^\top$. Therefore any vector belonging to $\text{span} \langle \mathbb{1} \rangle$ annihilates the right-hand side of (11) and therefore defines an equilibrium point of the DAE. There are no other equilibria because $\text{rk } \mathbb{L} = n - 1$.

Note that if $r = 1$, each one of the invariant spaces referred to in (a) amounts to a point, which actually belongs to the equilibrium line $y_1 = z_1 = z_2 = \dots = z_{n-1}$. In cases with $r \geq 2$, the transversality of the invariant spaces mentioned above and the equilibrium line follows easily from the fact that, in \mathbb{R}^n , the hyperplane $\sum_1^r y_i = k$ intersects transversally the line $y_1 = \dots = y_r = z_1 = \dots = z_{n-r}$. For later use it is worth emphasizing that the transversality of the intersection of the invariant spaces and the equilibrium line implies that each equilibrium is unique within each one of such invariant spaces.

(c) The dynamics can be examined in \mathbb{R}^r in terms of the y -variables via (12), having in mind that the z -components are given by (13). In particular, the invariant spaces and the line of equilibria discussed in (a) and (b) are projected respectively onto the hyperplanes of \mathbb{R}^r defined by

$$\sum_{i=1}^r y_i = k \tag{14}$$

and the line $\text{span} \langle \mathbb{1}_r \rangle$, that is

$$y_1 = y_2 = \dots = y_r. \tag{15}$$

Notice that both spaces are orthogonal to each other.

The rest of the proof relies on the fact that $-(\mathbb{L}_{11} - \mathbb{L}_{12} \mathbb{L}_{22}^{-1} \mathbb{L}_{21})$ is a symmetric, negative semidefinite, corank-one matrix. Its symmetry follows from that of \mathbb{L}_{11} and \mathbb{L}_{22} , together

with the identity $\mathbb{L}_{21} = \mathbb{L}_{12}^\top$ (recall that $\mathbb{L}_{ij} = A_i A_j^\top$). To check that it is negative semidefinite, write

$$-v^\top (\mathbb{L}_{11} - \mathbb{L}_{12} \mathbb{L}_{22}^{-1} \mathbb{L}_{21}) v = - \begin{pmatrix} v^\top & -(\mathbb{L}_{22}^{-1} \mathbb{L}_{21} v)^\top \end{pmatrix} \begin{pmatrix} \mathbb{L}_{11} & \mathbb{L}_{12} \\ \mathbb{L}_{21} & \mathbb{L}_{22} \end{pmatrix} \begin{pmatrix} v \\ -\mathbb{L}_{22}^{-1} \mathbb{L}_{21} v \end{pmatrix} \leq 0$$

for any $v \in \mathbb{R}^r$, since $-\mathbb{L} = -AA^\top$ is itself negative semidefinite. Similarly, the fact that $-(\mathbb{L}_{11} - \mathbb{L}_{12} \mathbb{L}_{22}^{-1} \mathbb{L}_{21})$ has corank one follows easily from the remark that

$$v \in \ker (\mathbb{L}_{11} - \mathbb{L}_{12} \mathbb{L}_{22}^{-1} \mathbb{L}_{21}) \Leftrightarrow \begin{pmatrix} v \\ -\mathbb{L}_{22}^{-1} \mathbb{L}_{21} v \end{pmatrix} \in \ker \begin{pmatrix} \mathbb{L}_{11} & \mathbb{L}_{12} \\ \mathbb{L}_{21} & \mathbb{L}_{22} \end{pmatrix}$$

together with the identity $\text{cork } \mathbb{L} = 1$.

Altogether, this implies that $-(\mathbb{L}_{11} - \mathbb{L}_{12} \mathbb{L}_{22}^{-1} \mathbb{L}_{21})$ has a simple zero eigenvalue, its associated eigenspace being the line of equilibria of (12), and $r-1$ real and negative eigenvalues; the corresponding eigenvectors are orthogonal to the equilibrium line because $-(\mathbb{L}_{11} - \mathbb{L}_{12} \mathbb{L}_{22}^{-1} \mathbb{L}_{21})$ is symmetric. Since the hyperplanes defined by (14) are themselves orthogonal to the equilibrium line, the evolution in such invariant hyperplanes is characterized by these $r-1$ negative eigenvalues.

This means that trajectories evolve exponentially towards the unique equilibrium in each invariant hyperplane; it is a trivial matter to check that this equilibrium must be defined by

$$y_i = \frac{y_1(0) + \dots + y_r(0)}{r}, \quad i = 1, \dots, r,$$

which is the unique solution to (14)-(15) with $k = y_1(0) + \dots + y_r(0)$. Finally, the identities

$$z_j = \frac{y_1(0) + \dots + y_r(0)}{r}, \quad j = 1, \dots, n-r,$$

follow from the fact that the equilibrium line spanned by $\mathbb{1}_r$ in \mathbb{R}^r is the y -projection of the line spanned by $\mathbb{1}$ in \mathbb{R}^n . \square

2.3 Constrained dynamics on general networks

The systematic analysis of constrained dynamics on general networks defines a topic of potential interest in many application fields. Just to name a few, the semiexplicit DAE (3) may, in neural learning processes, model an equilibrium assumption for the dynamics of the neural state variables (z in (3)), whereas the evolution of the synaptic weights (corresponding to the y -variables in (3)) characterizes the learning scheme; cf. [5, 46]. Beyond (3), constraints may arise not involving the algebraic variables z ; this yields a so-called Hessenberg DAE

$$y' = f(y, z) \tag{16a}$$

$$0 = g(y). \tag{16b}$$

Such DAEs may also arise naturally in the presence of constraints involving only node variables (standing for y in (16)) in models which also use certain variables supported on

the network edges (z in (16)). Along the lines later discussed in Section 3, Hessenberg DAEs may naturally arise also in multilevel networks in which certain restrictions among group variables are not visible at the agent's level, and viceversa. On the other hand, as it happens in nonlinear circuit theory (cf. [31, 45]), model reduction techniques in general networks may also lead to quasilinear DAEs of the form $A(x)x' = f(x)$, where $A(x)$ is an everywhere singular matrix-valued map. A reduction to an explicit ODE formulation may be difficult or impossible in these broader contexts. Besides circuit theory, where DAEs are now pervasive, different fields are starting to benefit from the systematic use of constrained formulations to describe network dynamics; examples are flow networks [27] (including flow dynamics in tree networks [37]), integrated process networks [15], dynamic traffic networks [21], pressurized water networks [48] or, from a different perspective, multiagent descriptor systems [51]. Analytical results directed to general networks are very promising in these and other related areas.

3 Dynamics on multilevel networks

We now turn our attention to a dynamical problem in a clustered network. As detailed below, we will assume that the set of agents (nodes) is partitioned into disjoint clusters or groups; state variables will be defined both on individual agents and on groups, and membership of a given group will entail a dynamical relation between the node and the group state variables. Additionally, we will assume that there are not only relations among individual agents, but also among groups. We are not concerned about the nature of these relations or the criterion according to which the nodes are clustered, but if it is of help the reader may think e.g. of a set of economic agents clustered in cooperatives, commercial transactions taking place both among agents and among cooperatives. Our goal is to analyze the dynamics of such a two-level network.

3.1 Two-level networks

We consider a two-level network defined by a 4-tuple (V, E, H, E_H) , where

- V is a set of $n \geq 1$ nodes or *agents*;
- E is a set of m *edges* connecting some pairs of nodes;
- H is a family of $p \geq 1$ non-empty sets or *groups* of nodes; and
- E_H is a set of q *generalized edges* connecting some pairs of groups.

Note that both (V, E) and (H, E_H) are graphs, whereas (V, H) defines a hypergraph (with hyperedges corresponding to the above-defined groups) if all nodes belong to at least one group [8]. This structure simply models a set of agents a) with a dyadic (binary) relation among them; b) joined together into certain groups; and c) displaying also a group-level

dyadic relation. Both (V, E) and (H, E_H) will be assumed to be simple graphs, and we will focus on cases in which H defines a *partition* of the set of agents into p pairwise-disjoint groups, the j -th one including $n_j \geq 1$ agents. Without loss of generality we assume that the agents are numbered according to this partition, so that the indices $1, \dots, n_1$ correspond to the first group, $n_1 + 1, \dots, n_1 + n_2$ to the second one, etc.

We assume that a reference direction is given to each edge and each generalized edge, giving both (V, E) and (H, E_H) a digraph structure. The incidence matrix describing the dyadic relation at the agents' level will be denoted by A , as in Section 2 (cf. (4)). Analogously, the incidence matrix describing the relation among groups will be written as $A_G = (a_{ij}) \in \mathbb{R}^{p \times q}$, where

$$a_{ij} = \begin{cases} 1 & \text{if generalized edge } j \text{ leaves group } i \\ -1 & \text{if generalized edge } j \text{ enters group } i \\ 0 & \text{if generalized edge } j \text{ is not incident with group } i. \end{cases} \quad (17)$$

In turn, the entries of the incidence matrix $A_H = (a_{ij}) \in \mathbb{R}^{n \times p}$ of the hypergraph (V, H) read as

$$a_{ij} = \begin{cases} 1 & \text{if the } i\text{-th agent belongs to the } j\text{-th group} \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

3.2 Node-group dynamics

In the setting considered above, dynamics may take place both at the agent and at the group level; therefore, state variables will be defined not only for agents but also for groups. For simplicity, we will assume that each agent and each group have exactly one (scalar) state variable, to be denoted by x_i for $i = 1, \dots, n$ and y_j for $j = 1, \dots, p$, respectively. Alone, the x - and the y -variables might be understood to correspond to dynamical processes defined on the graphs (V, E) and (H, E_H) . Certainly, the interest is placed on cases in which there is additionally a set of dynamic processes relating the agent state variables x_i with the group ones y_j .

In this context, we wish to analyze an extension of the redistribution scheme discussed in Section 2 to two-level networks. Specifically, both at the agent and at the group level there will be a dynamical process defined by the corresponding Laplacian matrices, that is, $\mathbb{L} = AA^\top$ and $\mathbb{G} = A_G A_G^\top$. Without an agent-group dynamic coupling, such dynamical systems would simply read as

$$x' = -\mathbb{L}x \quad (19)$$

and

$$y' = -\mathbb{G}y, \quad (20)$$

as in (7).

The coupling between agents and groups will be defined by the assumption that the group variable y_j stands for a collectively stored commodity at the j -th group, which (in the absence of transactions among agents or groups) evolves according to the individual amounts x_i stored by the agents which are clustered in that group. Specifically, we will assume that the rate at which the i -th agent contributes to y_j (or collects from y_j) is proportional to the difference

$$\frac{y_j}{n_j} - x_i.$$

The idea here is that the i -th agent contributes to (or collects from) y_j depending on the difference between its own amount of the resource, x_i , and the part y_j/n_j that would correspond to each agent in an eventual uniform distribution of the group stored commodity y_j to all n_j agents. This yields

$$x'_i = -x_i + \frac{y_j}{n_j}, \text{ for } i \text{ in group } j$$

and

$$y'_j = \sum_{i \text{ in } j} x_i - y_j, \quad j = 1, \dots, p.$$

The latter relation is derived immediately from the identity

$$y'_j = - \sum_{i \text{ in } j} x'_i,$$

where for the moment we are disregarding the flows among agents and among groups.

Note that, by construction, $A_H^\top A_H$ is a diagonal matrix of order p , the j -th diagonal entry being n_j , that is, the number of agents in the j -th group. The node-group dynamic relations described above can be then globally expressed as

$$x' = -x + A_H(A_H^\top A_H)^{-1}y \tag{21a}$$

$$y' = A_H^\top x - y, \tag{21b}$$

which can be understood as a dynamical process in the hypergraph (V, H) .

The whole dynamical process combines the flows among agents and among groups, described by (19) and (20), respectively, with the node-group dynamic relations defined by (21). This leads to the model

$$x' = -(\mathbb{I}_n + \mathbb{L})x + A_H(A_H^\top A_H)^{-1}y \tag{22a}$$

$$y' = A_H^\top x - (\mathbb{I}_p + \mathbb{G})y, \tag{22b}$$

where \mathbb{I}_n and \mathbb{I}_p are identity matrices of orders n and p .

3.3 A stability problem

System (22) describes a dynamical process in a two-level network, without any restriction so far in the topology of the graphs (V, E) and (H, E_H) . We address here a stability problem in a simplified setting; we assume that there are no interactions at the agents' level or, in graph-theoretic terms, that all nodes are disconnected. The flow (20) describing the interactions among groups in the graph (H, E_H) and the node-group dynamics (21) in the hypergraph (V, H) still apply, to yield the model:

$$x' = -x + A_H(A_H^\top A_H)^{-1}y \quad (23a)$$

$$y' = A_H^\top x - (\mathbb{I} + \mathbb{G})y. \quad (23b)$$

Below we denote by $\chi(i)$ the group to which agent i belongs.

Theorem 2. *Consider system (23) and assume that the graph (H, E_H) , describing the relations among groups, is connected.*

(a) *System (23) has a line of equilibria defined by the relations*

$$x_i = \frac{y_{\chi(i)}}{n_{\chi(i)}} \quad (i = 1, \dots, n), \quad y_1 = y_2 = \dots = y_p. \quad (24)$$

(b) *An initial condition $(x(0), y(0))$ converges exponentially to the equilibrium defined by*

$$x_i = \frac{y_{\chi(i)}}{n_{\chi(i)}}, \quad i = 1, \dots, n, \quad y_j = \frac{1}{2p} \left(\sum_{i=1}^n x_i(0) + \sum_{k=1}^p y_k(0) \right), \quad j = 1, \dots, p. \quad (25)$$

Proof.

(a) By means of a Schur reduction of (23), equilibria are easily checked to satisfy

$$A_H^\top A_H (A_H^\top A_H)^{-1}y - (\mathbb{I} + \mathbb{G})y = 0,$$

that is,

$$\mathbb{G}y = 0.$$

Since the graph (H, E_H) is connected, $\mathbb{G} = A_G A_G^\top$ is rank-deficient by one, and $\ker \mathbb{G} = \ker A_G^\top$ is defined by the relations $y_1 = y_2 = \dots = y_p$. The x -components are defined by $x = A_H(A_H^\top A_H)^{-1}y$. By construction $A_H^\top A_H$ is a diagonal matrix, and the j -th diagonal

equals n_j , that is, the number of agents in the j -th group; we then have

$$A_H = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & 0 & 0 & 0 & \dots & 0 \\ (n_1) & 0 & 0 & 0 & \dots & 0 \\ \vdots & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & \vdots & 0 & 0 & \dots & 0 \\ 0 & (n_2) & 0 & 0 & \dots & 0 \\ 0 & \vdots & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad (A_H^\top A_H)^{-1} = \begin{pmatrix} \frac{1}{n_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{n_2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{n_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{n_p} \end{pmatrix} \quad (26)$$

so that $x = A_H(A_H^\top A_H)^{-1}y$ amounts to $x_i = \frac{y_{\chi(i)}}{n_{\chi(i)}}$ for $i = 1, \dots, n$, as stated in (24).

(b) System (23) will be proved stable by using Geršgorin theorem [26], according to which all eigenvalues of the coefficient matrix of the right-hand side of (23), namely

$$M = \begin{pmatrix} -\mathbb{I} & A_H(A_H^\top A_H)^{-1} \\ A_H^\top & -(\mathbb{I} + \mathbb{G}) \end{pmatrix},$$

must lie on the union of the discs $|z - m_{jj}| \leq R_j$, with

$$R_j = \sum_{\substack{i=1 \\ i \neq j}}^{n+p} |m_{ij}|.$$

We are denoting by m_{ij} the entries of the matrix M . It is obvious that

$$m_{jj} = -1 \text{ if } 1 \leq j \leq n,$$

and

$$m_{jj} = -1 - d_{j-n} \text{ if } n+1 \leq j \leq n+p,$$

where d_j stands for the number of connections of the j -th group to other groups (that is, its degree in the graph (H, E_H)); we are making use of the fact that \mathbb{G} is the Laplacian matrix of the graph (H, E_H) describing the group connections.

On the other hand, there is only one non-vanishing entry (with value 1) in any column of A_H^\top , so that

$$R_j = 1 \text{ if } 1 \leq j \leq n.$$

This means that n discs are centered at -1 and have radius 1. Additionally, by construction one can see that the sum of the entries in any column of $A_H(A_H^T A_H)^{-1}$ is 1; indeed, from (26) we may check that the j -th column of $A_H(A_H^T A_H)^{-1}$ has n_j nonvanishing entries equal to $1/n_j$. Together with the fact that the off-diagonal entries of $-\mathbb{G}$ define the adjacency matrix of the graph (H, E_H) , this yields

$$R_j = 1 + d_{j-n} \text{ for } n+1 \leq j \leq n+p$$

so that the remaining p discs are centered at $-1 - d_{j-n}$ and have radius $1 + d_{j-n}$ (with $j = n+1, \dots, n+p$). For later use, notice that this also shows that the sum of the entries of any column of M does vanish.

The remarks above prove that all Geršgorin discs are located on the left half complex plane, except for a tangency with the imaginary axis at the origin. From (a) it follows that the matrix M has a unique zero eigenvalue (with eigenvectors in $\ker \mathbb{G}$), so that the other $n+p-1$ ones are away from the imaginary axis.

The last remark needed to prove (b) is that the quantity $\sum_{i=1}^n x_i + \sum_{j=1}^p y_j$ is preserved along trajectories, which is a consequence of the aforementioned fact that the sum of the entries in each column of M is null (so that $\sum_{i=1}^n x'_i + \sum_{j=1}^p y'_j = 0$). This means that

$$\sum_{i=1}^n x_i + \sum_{j=1}^p y_j = \sum_{i=1}^n x_i(0) + \sum_{j=1}^p y_j(0)$$

is a family of invariant hyperplanes, the evolution rate in each one of them being characterized by the $n+p-1$ eigenvalues which are located on the left half-plane. Therefore, the dynamics evolves exponentially towards the point defining the intersection of such invariant planes and the line of equilibria arising in (a). By combining the identities $y_1 = y_2 = \dots = y_p$ with the fact that the conditions $x_i = y_{\chi(i)}/n_{\chi(i)}$ yield $\sum_{i=1}^n x_i = \sum_{j=1}^p y_j$ at equilibrium, it is easy to check that this intersection is given by (25) and the proof is complete. \square

Theorem 2 extends the results discussed in Section 2 to the two-level dynamic network here considered. Now the scheme converges asymptotically to an equal distribution of the resource among all groups, and to a uniform distribution among the individual agents in each group. Differences may arise between agents of different groups.

We leave for future study the full analysis of (22), which may display more intricate dynamics; further research may extend these results to problems with flowrates different from the ones yielding the graph Laplacian matrices in (19) and (20), or with node-group dynamic relations different from (21). Our results should also be of interest in an eventual deterministic, continuous-time dynamic analysis of multilevel networks in general, possibly accommodating more aggregation levels.

4 On the equilibrium set of potential-driven flow networks

Theorems 1 and 2 above share a somewhat unusual property in dynamical systems theory, namely the existence of a line of equilibria. Non-isolated equilibria have received attention within the theory of bifurcation without parameters (cf. [33] and references therein; see also [4]). In our context, the key element supporting the existence of a line of equilibria is the fact that systems (11) and (23) essentially describe the dynamics of potential-driven flows. As detailed later, equilibria define a one-dimensional manifold (a line) because of the implicit assumption that the networks involved are positively weighted. Without this assumption, the structure of the equilibrium set may be more intricate; in this section we examine such structure in flow networks without this positiveness assumption, arriving at a graph-theoretic characterization of the structure of the equilibrium set in terms of the network spanning trees. In particular, we will apply our results to networks based on signed graphs, originally introduced by Harary [22].

4.1 Nonlinear flows

A potential-driven flow on a (directed) graph with n nodes and m edges is defined by a dynamical system of the form

$$\alpha_i x'_i = -A_i f(A^\top x), \quad i = 1, \dots, n. \quad (27)$$

Here $x = (x_1, \dots, x_n)$ and A_i is the i -th row of the incidence matrix A as defined in (4). In turn $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a possibly nonlinear, differentiable map describing the flowrates in the edges. It is assumed to have a diagonal structure, that is

$$f = f_1 \times \dots \times f_m,$$

with $f_j : \mathbb{R} \rightarrow \mathbb{R}$ depending only on $(A^\top)_j x$, where $(A^\top)_j$ is the j -th row of A^\top ; this means that the flowrate in edge j (connecting nodes i and k) depends only on the difference $x_i - x_k$. The variables x can be thought of as a potential and hence the “potential-driven” label for the flow: electric potential or pressure are examples in electrical circuits and water networks, respectively. Finally, the coefficients α_i are 0 or 1 for each node; the case $\alpha_i = 1$ (resp. $\alpha_i = 0$) corresponds to nodes which accumulate (resp. do not accumulate) a certain amount of the quantity or resource (e.g. electrical charge, water, gas, a given commodity, etc.) which is flowing in the network; note that (27) describes the continuity equations at both types of nodes.

A simple example with $\alpha_i = 1$ for all nodes, f being the identity map, is defined by (5)-(6), yielding the Laplacian dynamics (7). A problem combining nodes with $\alpha_i = 1$ and $\alpha_i = 0$ is given by system (11), analyzed in Section 2. Additionally, the reader may think of a (possibly nonlinear) resistive circuit as an example with $\alpha_i = 0$ for all nodes.

4.2 Equilibria and the subimmersion theorem

To make things simpler, in the sequel we assume that $\alpha_i = 1$ for all nodes, focusing on dynamical systems of the form

$$x' = -Af(A^\top x), \quad (28)$$

the aforementioned restrictions on the form of f still holding. Also for the sake of simplicity we assume that the digraph is connected, so that $\text{rk } A = n - 1$. For notational convenience we denote by $F(x)$ the right-hand side of (28), that is,

$$F(x) = -Af(A^\top x). \quad (29)$$

It is easy to check that equilibria of (28) may never be isolated; indeed, provided that $F(x)$ vanishes for a given x^* , then $x = x^* + v$ also annihilates $F(x) = -Af(A^\top x)$, for any $v \in \ker A^\top$. Note that this kernel is never trivial, being one-dimensional in a connected digraph. The problem we address in this section is the characterization of the cases in which the equilibrium set is locally a line, as it happens in Theorems 1 and 2. We show below that the one-dimensional nature of the equilibrium set in these theorems is implicitly supported on a positiveness assumption on the digraph weights, without which the equilibrium set may locally have a higher dimension. In general, we do not require f to be linear; when f is a linear map the equilibrium set is a linear manifold and the results hold globally.

We will make use of the subimmersion theorem, which states that if Ω is an open subset of \mathbb{R}^n , and F is a smooth mapping $\Omega \rightarrow \mathbb{R}^p$ such that the Jacobian matrix $F'(x)$ has constant rank $r \leq p$ on Ω , then for every y in $F(\Omega)$ the set $F^{-1}(y)$ is a submanifold of Ω with dimension $n - r$ (see e.g. Th. 3.5.17 in [1] or Th. III.5.8 in [10]). The result also holds if F has constant rank r on a neighborhood of $F^{-1}(y)$. We will use this result with F given in (29), $y = 0$, $p = n$ and $r = n - 1$ to characterize the problems in which the equilibrium set is locally a line around a given equilibrium point x^* . The Jacobian matrix $F'(x)$ reads as

$$F'(x) = -Af'(A^\top x)A^\top \quad (30)$$

and, since $\text{rk } A = n - 1$, it follows that $F'(x)$ is persistently rank deficient. Therefore, for the equilibrium set of (28) to be one-dimensional (at least locally around x^*), it is enough to derive conditions guaranteeing $\text{rk } F'(x^*) = n - 1$, since this maximum rank would necessarily be attained also on a neighborhood of x^* . In order to examine the rank of $F'(x^*)$, let us denote the derivatives of the components of f at x^* as

$$W_j = f'_j(A^\top x^*), \quad j = 1, \dots, m, \quad (31)$$

and let W stand for the diagonal matrix with entry W_j in the j -th diagonal position. Note that this matrix is indeed diagonal because of the assumption that f_j depends only on $(A^\top)_j x$. With this notation we have

$$F'(x^*) = -AWA^\top. \quad (32)$$

This expression shows that the Jacobian matrix $F'(x^*)$ has the structure of a weighted Laplacian matrix, with weights being defined by the derivatives $f'_j(A^\top x^*)$.

4.3 Positive weights

If all weights W_j (that is, all derivatives $f'_j(A^\top x^*)$) are positive, then it is a simple matter to check that

$$\ker AWA^\top = \ker A^\top. \quad (33)$$

Indeed, just note that $AWA^\top u = 0$ implies $u^\top AWA^\top u = 0$ and therefore $A^\top u = 0$ because of the positiveness of the diagonal matrix W . The relation depicted in (33) implies that

$$\text{rk } F'(x^*) = \text{rk } AWA^\top = \text{rk } A^\top = n - 1$$

and, because of the subimmersion theorem, it follows that equilibria actually define a curve near x^* .

Implicitly, this underlies the existence of a line of equilibria in the setting of Theorems 1 and 2. Note that system (7), defined by the Laplacian matrix AA^\top , can be understood as a particular instance of the product AWA^\top with $W = \mathbb{I}_m$. Actually, one may show that the results of Sections 2 and 3 actually hold if the product AA^\top is replaced by AWA^\top with a positive, diagonal W ; in other words, those results still apply if the networks are assumed to be positively weighted. Note that the linear setting considered in those sections yields a linear manifold of equilibria and avoids the need for a local approach.

4.4 Negative weights and the structure of the equilibrium set

If W in (32) includes negative entries, the remarks just stated do not apply, and we may actually find potential-driven flow dynamics on connected graphs displaying higher (≥ 2) dimensional manifolds of equilibria. This may be the case even in linear problems: examples in a linear context, involving signed graphs, can be found in subsection 4.5 below.

A different approach is needed to analyze the structure of the equilibrium set in digraphs with possibly negative weights. In this setting, we will drive the problem to a context known in the framework of circuit theory, namely the analysis of nodal admittance matrices, which can be performed along the topological approach stemming from the work of J. C. Maxwell [14, 44]. In Theorem 3, which is the main result of this section, we make use of the notion of the weight of a spanning tree; this is simply the product of the weights of the tree edges.

Theorem 3. *Assume that the dynamical system (28) is defined on a connected digraph. The Jacobian matrix $F'(x^*)$ in (32) has corank one if and only if the sum of the spanning tree weights does not vanish, where the edge weights are given by (31). If this sum is not null, then the set of equilibria is a curve locally around the equilibrium point x^* .*

Proof. The matrix AWA^\top in the right-hand side of (32) is a weighted Laplacian matrix, which is known to be rank deficient since $\text{rk } A = n - 1$. The first step in the proof shows that all $(n - 1)$ -minors of AWA^\top do vanish if and only if a single one of them does; therefore, the identity $\text{rk } AWA^\top = n - 1$ will hold if and only if one (hence any) principal minor does not vanish.

By construction, minors of AWA^\top of order $n-1$ are determinants of products of the form

$$A_{r_1}WA_{r_2}^\top,$$

where A_{r_1} and A_{r_2} are *reduced* incidence matrices, defined by two arbitrary choices of $(n-1)$ rows of A . The key remark here comes from a graph-theoretic property, saying that any set of $n-1$ rows of the incidence matrix of a connected digraph are linearly independent (actually defining a basis of the so-called *cut space*; cf. [9]). This means that a relation of the form

$$A_{r_2} = KA_{r_1}$$

holds for a non-singular matrix K . Therefore

$$\det(A_{r_1}WA_{r_2}^\top) = \det(A_{r_1}WA_{r_1}^\top) \det K$$

and it follows that $\det(A_{r_1}WA_{r_2}^\top)$ does vanish if and only if $\det(A_{r_1}WA_{r_1}^\top)$ does. Hence, the eventual vanishing of all $(n-1)$ -minors occurs if and only if a single one of them is null; equivalently, in order to examine the condition $\text{rk } AWA^\top = n-1$ it suffices to study the vanishing of a principal minor, that is, the condition

$$\det(A_rWA_r^\top) \neq 0, \tag{34}$$

for an arbitrary choice of a reduced incidence matrix A_r . Without loss of generality, in what follows we are allowed to work e.g. with the reduced incidence matrix A_r defined by the first $n-1$ rows of A . The corank-one condition on $F'(x^*)$ then amounts to evaluating the condition (34).

This can be performed using a Cauchy-Binet determinantal expansion of (34) (cf. [26]), which yields

$$\det(A_rWA_r^\top) = \sum_{\alpha, \beta} \det(A_r^\alpha) \det(W_\alpha^\beta) \det((A_r^\top)_\beta), \tag{35}$$

the sum being taken over all index sets $\alpha, \beta \subseteq \{1, \dots, m\}$ with $n-1$ elements. In the submatrices involved in this expansion, the subscripts (resp. superscripts) α, β are used to select a set of rows (resp. columns); for instance, A_r^α denotes the submatrix of A_r specified by all rows and the columns indexed by α , whereas W_α^β is the submatrix of W defined by the rows and columns specified by α and β , respectively.

The expansion depicted in (35) can be simplified using the following two remarks. First, a set of columns of the reduced incidence matrix A_r is known to yield a non-singular matrix if and only if such columns correspond to the edges of a spanning tree; in that case, we have $\det(A_r^\alpha) = \pm 1$, because incidence matrices are totally unimodular (see e.g. [3]).

Second, because of the diagonal form of W , one can easily check that $\det(W_\alpha^\beta)$ does vanish if $\alpha \neq \beta$; when $\alpha = \beta$, we have

$$\det(W_\alpha^\alpha) = \prod_{j \in \alpha} W_j,$$

which is the tree weight when α specifies a spanning tree. Note additionally that, if $\alpha = \beta$ then $\det(A_r^\alpha) = \pm 1$ equals $\det((A_r^\top)_\beta) = \det((A_r^\top)_\alpha)$, so that

$$\det(A_r^\alpha) \det((A_r^\top)_\alpha) = 1.$$

Altogether, these remarks prove that the determinantal expansion displayed in (35) amounts to

$$\det(A_r W A_r^\top) = \sum_{\alpha} \prod_{j \in \alpha} W_j,$$

where the sum is taken over the index sets α which specify a spanning tree. It follows that any $(n-1)$ -minor of the Jacobian matrix $F'(x^*)$ is non-null, and therefore $F'(x^*)$ has corank one, if and only if the sum of the spanning tree weights does not vanish, as claimed.

Provided that the non-vanishing condition (34) holds, the identity $\text{rk } F'(x^*) = n - 1$ yields, as a direct consequence of the subimmersion theorem in the terms stated above, a local one-dimensional structure for the equilibrium set near x^* , and the proof is complete. \square

Theorem 3 provides a graph-theoretic characterization, in terms of the digraph tree structure, of the problems which systematically lead to curves (and not higher dimensional or singular manifolds) of equilibria in potential-driven flow networks with (possibly) some negative weights. Stability aspects in this context are in the scope of future research; note in particular that the failing of the non-vanishing requirement in the sum of tree weights might be responsible for bifurcation phenomena in nonlinear problems.

Certainly, if all weights are positive then the sum arising in Theorem 3 is positive and therefore non-null, because all tree weights are positive. This is of course consistent with the discussion of subsection 4.3 regarding the one-dimensional nature of the equilibrium set in positively weighted networks, for which there is no need to use these tree-based tools.

4.5 Signed graphs

A nice corollary of Theorem 3 holds for signed graphs, originally introduced by Harary [22] and widely used in balance theory (cf. [23] and more recent references such as [11, 16]). A *signed graph* or *s-graph* is a graph (V, E) endowed with a map $E \rightarrow \{-1, 1\}$, that is, an assignment of either a $+1$ or a -1 weight to all m edges. As above, we will let $W \in \mathbb{R}^{m \times m}$ stand for the diagonal matrix of weights.

In this context, a flow problem which arises as a particular case of (28) is

$$x' = -AWA^\top x. \tag{36}$$

This system can be again understood as a flow, via the continuity equations $x' = -Au$, in which the flowrates u are given by the linear relation $WA^\top x$. With respect to (5)-(6), yielding the Laplacian dynamics (7), now the presence of -1 weight values models a flow in

which adjacent agents i, k tend to *increase* the difference between x_i and x_k , since the flow from i to k now equals $x_k - x_i$ (instead of $x_i - x_k$, as in subsection 2.1).

Contrary to the results in Sections 2 and 3, in which equilibria define a line, the presence of negative weights might now result in higher dimensional equilibrium manifolds. The cases in which this may happen are exactly characterized in Corollary 1 below. Since all weights are $+1$ or -1 , we may now define a spanning tree as *positive* or *negative* simply if its weight product is $+1$ or -1 , respectively, or, equivalently, if it contains an even (resp. odd) number of edges with negative weight.

Corollary 1. *Let the dynamical system (36) be defined on a connected signed digraph. Then, the dimension of the equilibrium set is higher than one if and only if the numbers of positive and negative spanning trees coincide.*

This results follows immediately from Theorem 3 since, for the sum of weight products to vanish in a signed graph, the amount of positive trees (which are responsible for a $+1$ term in the sum) must obviously match the number of negative trees (which yield a -1 term in the sum).

Examples. Simple examples illustrating the result above can be defined using the graphs shown in Figure 1. The first case, on the left of the figure, is simply a 4-cycle with two positive signs (continuous lines) and two negative signs (dashed lines). This graph has just four spanning trees, two of which are positive and the other two negative. The second example, on the right, depicts a complete graph K_4 with two positive and four negative signs; this is a so-called balanced structure [22], in which the nodes may be split in two groups in a way such that all edges inside a group have positive weights and all connections between groups are negative. In this example the groups are defined by the two nodes on the left and those on the right, respectively. According to Cayley's formula this graph has $4^2 = 16$ spanning trees, and it is not difficult to check that exactly half of them are positive. This means that the dynamical system (36) should exhibit in both cases a linear manifold of equilibria with dimension greater than one.

In the first case, we number and orientate each edge beginning on the top and according to a clockwise orientation of the cycle. This yields

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

With $W = \text{diag}(1, 1, -1, -1)$, the right-hand side of (36) reads in this case as

$$-AWA^T = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix},$$

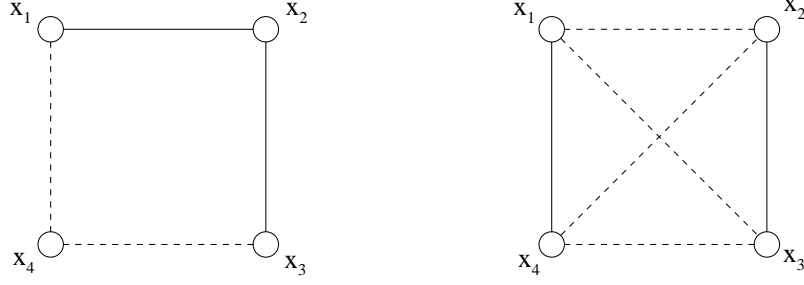


Figure 1: Sign assignments yielding degenerate flow dynamics on (a) a 4-cycle; (b) K_4 . Edges with a negative sign are dashed.

The kernel of this matrix defines the equilibrium set and is defined by the relations

$$x_2 = x_4 = \frac{x_1 + x_3}{2},$$

hence defining a two-dimensional linear manifold (a plane) of equilibria. Notice that these equilibrium points arise from a constant flow $x_i - x_{i+1}$ ($i = 1, \dots, 4$, with the terminological abuse $x_5 = x_1$) which annihilates all derivatives x'_i , flowing in the clockwise (resp. counter-clockwise) direction if $x_1 > x_3$ (resp. if $x_1 < x_3$). Note that the equilibrium plane includes the line $x_1 = x_2 = x_3 = x_4$ for which the flow vanishes.

In the example defined on K_4 (on the right of Figure 1), we number and orientate the edges according to the following incidence matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 \end{pmatrix},$$

and the weights $W = \text{diag}(-1, 1, -1, 1, -1, -1)$ yield

$$-AWA^\top = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Now the equilibrium set is three-dimensional, being defined by the identity

$$x_1 + x_4 = x_2 + x_3. \quad (37)$$

Again, these equilibrium solutions yield nonvanishing flows in the graph edges, except for those in the line $x_1 = x_2 = x_3 = x_4$. This example also illustrates that the rank drop in the product AWA^\top may be greater than two.

We leave for future work the analysis of the nature of such equilibrium solutions exhibiting non-vanishing flows, that is, equilibria which are not in the line $x_1 = x_2 = \dots = x_n$; this analysis should explain, for instance, the relation of the equilibrium solutions in the second example with the presence of a balanced structure in the graph: note that (37) expresses that the total amount of the resource x stored in both groups (namely, $\{x_1, x_4\}$ and $\{x_2, x_3\}$) is the same. More generally, further research might address stability aspects and bifurcations in these negatively weighted networks, not only in signed graphs but specially in the nonlinear setting introduced in subsection 4.1.

Note finally that even though in Section 4 we have focused for simplicity on flow networks with homogeneous agents (because of the assumption $\alpha_i = 1$ for all nodes, cf. subsection 4.2), the results are also of potential interest in broader contexts, involving e.g. heterogeneous nodes or group dynamics as in Sections 2 and 3.

5 Concluding remarks

We have addressed in this paper some qualitative problems involving network dynamics beyond the scenario defined by a set of dynamical systems supported on the nodes of a graph. First, the attention has been focused on potential-driven flow dynamics either on networks with heterogeneous agents or on multilevel networks. Differential-algebraic models, arising here in flow networks with heterogeneous agents, are worth receiving further attention in order to model and analyze constrained dynamics on general networks. Dynamical systems on multilevel networks also have a broad scope for future research. From a different perspective, we have analyzed the structure of the equilibrium set in (possibly nonlinear) flow networks with negative weights. The results in this regard apply in particular to signed graphs, and the connection of our results to the theory of balance and clusterability in signed graphs requires further analysis. In greater generality, stability properties and bifurcations are worth being studied in networks with negative weights.

Our research is of potential interest in the analysis of social and economic networks involving flows. Future work might also extend these results to problems displaying network evolution, and also to stochastic and/or discrete-time contexts.

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